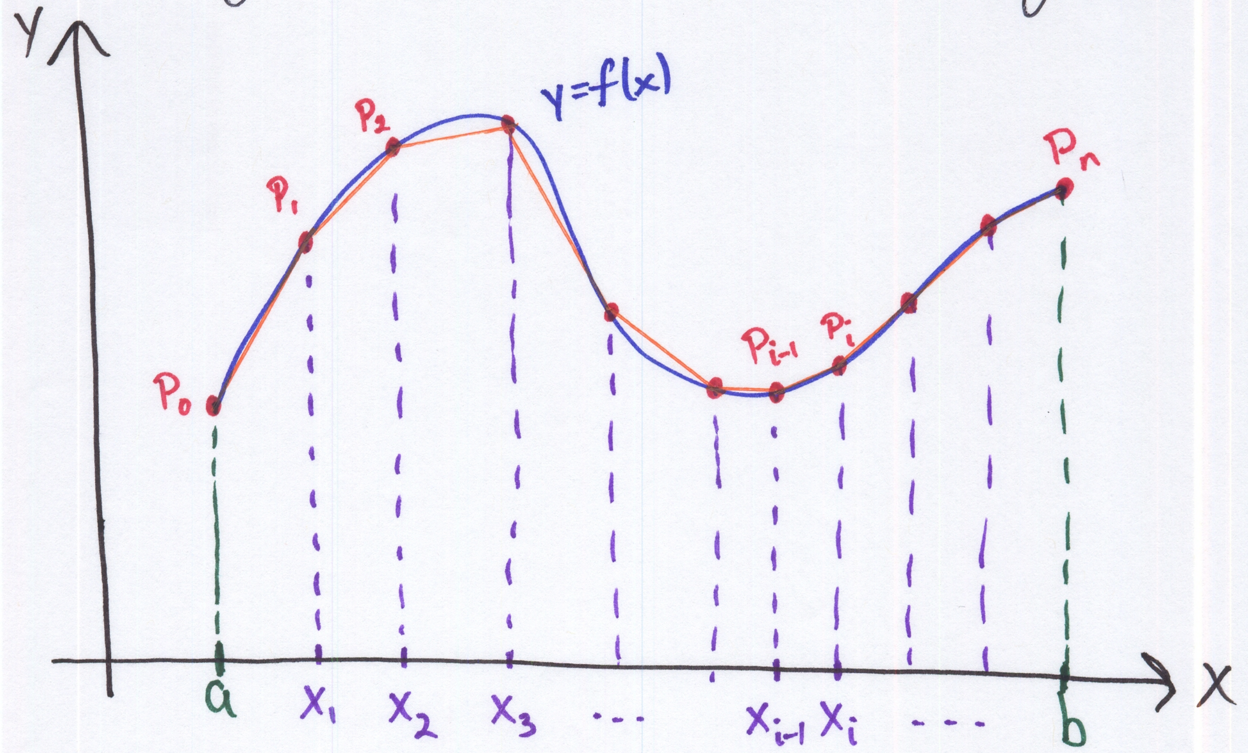
Lecture 16

8.1 - Arc Length

Suppose we have a curve $y=f(x)$ on an interval $[a,b]$, and that $f(x)$ is differentiable on $[a,b]$ and $f'(x)$ is continuous on $[a,b]$. We would like to find the length of this curve. We approximate the curve by straight lines and add up their lengths. In the limit, this gives the actual curve length:



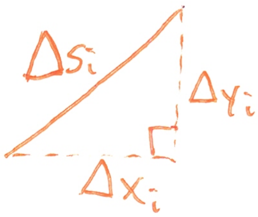
Thus, we get the length as:

$$\boxed{L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|}$$

($|P_{i-1}P_i|$ = distance from P_{i-1} to P_i)

If we call the length of the line segment between P_{i-1} and P_i : Δs_i, then

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$



where Δx_i & Δy_i are the respective changes in x & y. Doing a clever factorization to Δs_i :

$$\Delta s_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i$$

This gives us the definition of arc length:

$$L = \int_{\text{curve}} ds = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \right) \Delta x_i$$

(Calc III notation)

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

And, from this comes the definition of the arclength function:

$$\boxed{s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt}$$

this gives the length starting at $(a, f(a))$ and ending at $(x, f(x))$.

Ex: Find the length of the curve $y = 1 + 6x^{3/2}$ for $0 \le x \le 1$.

Sol: $\frac{dy}{dx} = 9x^{1/2}$

$$L = \int_0^1 \sqrt{1 + (9x^{1/2})^2}\, dx = \int_0^1 \sqrt{1+81x}\, dx \quad \begin{pmatrix} u = 1+81x \\ du = 81\,dx \end{pmatrix}$$

$$= \frac{1}{81}\int_1^{82} u^{1/2}\, du = \frac{1}{81}\left(\frac{2}{3}u^{3/2}\Big|_1^{82}\right) = \boxed{\frac{2}{243}\left(82^{3/2} - 1\right)}$$

Sometimes, a curve is more easily described by $x = g(y)$, $c \le y \le d$, where g is differentiable on $[c,d]$ and g' is continuous there as well (an easier way to say this is that g is C^1 on $[c,d]$). Then, by factoring Δy_i out of Δs_i instead of Δx_i, we get another formula for the arclength:

$$\Delta s_i = \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\, \Delta y_i$$

$$\rightsquigarrow \boxed{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy}$$

Ex: Find the arclength function, $s(y)$, for the curve $x = \frac{y^4}{8} + \frac{1}{4y^2}$, starting at $(\frac{3}{8}, 1)$ (and increasing in y-values).

Sol: $\frac{dx}{dy} = \frac{y^3}{2} - \frac{1}{2y^3}$

$$s(y) = \int_1^y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dt = \int_1^y \sqrt{1 + \left(\frac{t^3}{2} - \frac{1}{2t^3}\right)^2}\, dt$$

$$= \int_1^y \sqrt{1 + \frac{t^6}{4} - \frac{1}{2} + \frac{1}{4t^6}}\, dt = \int_1^y \sqrt{\frac{t^6}{4} + \frac{1}{2} + \frac{1}{4t^6}}\, dt$$

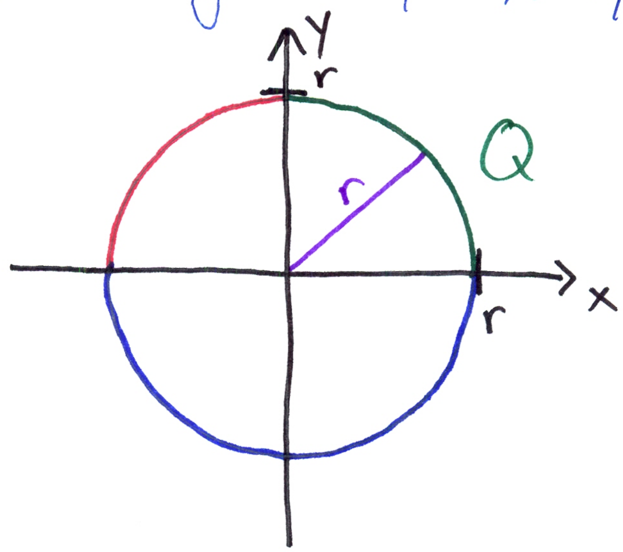
$$= \int_1^y \sqrt{\left(\frac{t^3}{2} + \frac{1}{2t^3}\right)^2}\, dt = \int_1^y \left(\frac{t^3}{2} + \frac{1}{2t^3}\right) dt = \left(\frac{t^4}{8} - \frac{1}{4t^4}\right)\Big|_1^y$$

$$= \left(\frac{y^4}{8} - \frac{1}{4y^4}\right) - \left(\frac{1}{8} - \frac{1}{4}\right) = \boxed{\frac{y^4}{8} - \frac{1}{4y^4} + \frac{1}{8}}$$

It is possible, and, in fact, is usually the case, that these arclength integrals are difficult or impossible to compute. In these cases we use an approximation method, such as Simpson's rule.

Ex: Show that the circumference of a circle of radius r is in fact $2\pi r$.

Sol: This circle is given by $x^2+y^2=r^2$.



We need to find $\frac{dy}{dx}$. To do this, we need to solve for y: $y=\pm\sqrt{r^2-x^2}$. If we take $y=\sqrt{r^2-x^2}$, we only get the top half of the circle, but that's okay since the length of the whole circle is twice that of the top half, by symmetry. Then

$$y' = (-2x)\cdot\frac{1/2}{\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}$$

We could integrate from $-r$ to r, but integrating from 0 to r and doubling that gives us the length of the top half. Call the quarter circle Q, then:

Circumference of circle = 4* length of Q

$$= 4\int_0^r \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx = 4\int_0^r \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$= 4\int_0^r \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx = 4\int_0^r \sqrt{\frac{r^2}{r^2-x^2}}\,dx = 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx$$

$x = r\sin\theta$
$dx = r\cos\theta\, d\theta$

$$= 4\int_0^{\pi/2} \frac{r}{\sqrt{r^2-r^2\sin^2\theta}}(r\cos\theta\, d\theta) = 4\int_0^{\pi/2} \frac{r^{\cancel{2}}\cancel{\cos\theta}}{\cancel{r}\,\cancel{\cos\theta}}\,d\theta = 4\int_0^{\pi/2} r\,d\theta$$

$$= 4r\theta\Big|_0^{\pi/2} = 4r\left(\frac{\pi}{2}\right) = \boxed{2\pi r}$$